Using Physical Informed Neural Network (PINN) to Improve a $k - \omega$ Turbulence Model [2]

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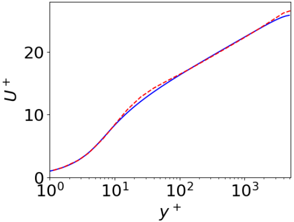
The Wilcox $k - \omega$ turbulence model reads [4]

$$
\begin{aligned}
\frac{\partial \bar{v}_i}{\partial x_i} &= 0 \\
\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_t)\frac{\partial \bar{v}_i}{\partial x_j}\right] \\
\frac{\partial \bar{v}_j k}{\partial x_j} &= P^k + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] - C_\mu k \omega \\
\frac{\partial \bar{v}_j \omega}{\partial x_j} &= C_{\omega_1}\frac{\omega}{k}P^k + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] - C_{\omega 2}\omega^2 \\
P^k &= \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j}, \quad \nu_t = \frac{k}{\omega}
\end{aligned}
\tag{1}
$$

The standard coefficients are used, i.e. $C_{\omega 1} = 5/9$, $C_{\omega 2} = 3/40$, $\sigma_k = \sigma_\omega = 2$ and $C_\mu = 0.09$.

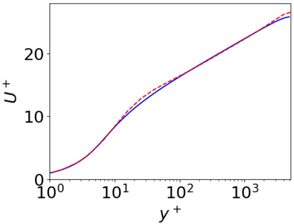


(A) Mean velocity.

FIGURE: Fully-developed channel flow. Solid lines: $k-\omega$; dashed lines:DNS [3].

- The mean flow, shear stress (and hence the turbulent viscosity, ν_t) agree well

Fully-developed channel flow, $k-\omega$ model, $Re_\tau = 5\,200$



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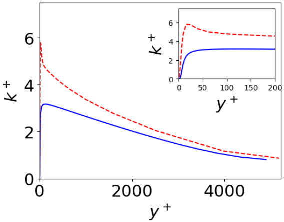

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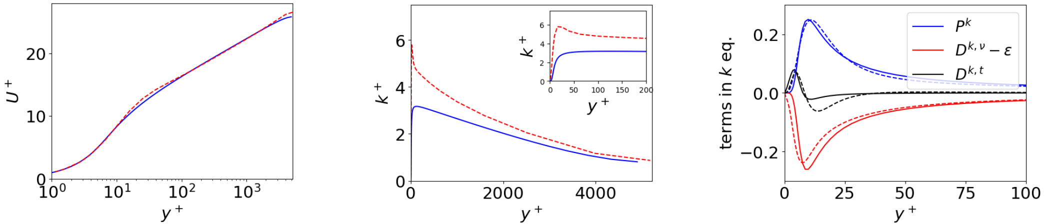

(A) Mean velocity. (B) Turb. kinetic energy. (C) Terms in k equation.

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- The mean flow, shear stress (and hence the turbulent viscosity, ν_t) agree well
- But not the turbulent, kinetic energy
- It seems to be because the diffusion of k is poorly predicted

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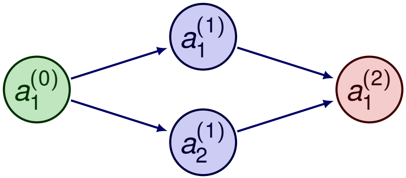
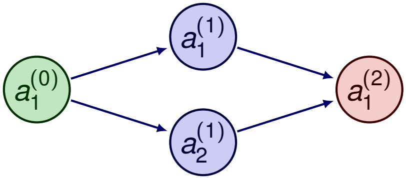
- $\nu_{t,k} = \nu_{t,NN}$ in Eq. 2, will be predicted by PINN while minimizing the error Q^2.



- I create a NN that finds a damping function, $Y \equiv f$, as a function of input $X \equiv y^+$
- 1 input ($X = a_1^{(0)}$), 1 hidden layer with 2 neurons ($a_1^{(1)}, a_2^{(1)}$) and 1 output ($Y = a_1^{(2)}$)



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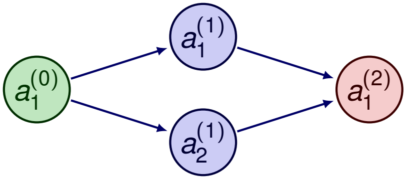
class NN(nn.Module):
 def super-__init__(self):
  self.layer_1=nn.Linear(1, 2) # Connection 0-1
  self.layer_2=nn.Linear(2, 1) # Connection 1-2
 def forward(self, x):
  y = torch.nn.functional.sigmoid(self.layer_1(x)) # a_1^{(1)}, a_2^{(1)}, hidden-layer
  output = torch.nn.functional.sigmoid(self.layer_2(y)) # a_1^{(2)}, output-layer
```



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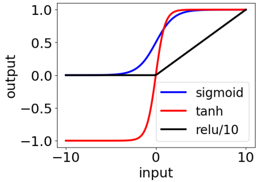


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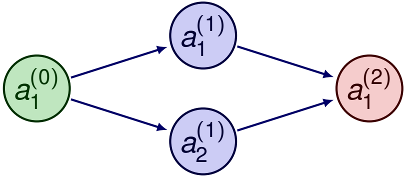
NEURAL NETWORK (NN). FORWARD



Activation 1: $$a_1^{(1)} = s_1^{(1)}\left(w_1^{(0)}a_1^{(0)} + b_1^{(0)}\right)$$

Activation 2: $$a_2^{(1)} = s_2^{(1)}\left(w_2^{(0)}a_1^{(0)} + b_2^{(0)}\right)$$

Output: $$a_1^{(2)} = s_1^{(2)}\left(w_1^{(1)}a_1^{(1)} + b_1^{(1)} + w_2^{(1)}a_2^{(1)} + b_2^{(1)}\right) \equiv Y$$

- s is an activation function (`linear`, `sigmoid`, `tanh`, ...)

The Python code for the simple NN model is given in the listing below

```
#  initiate the NN model
model = NN()
# define input, X
X=np.zeros(nj,1))
X[:,0] = scaler_yplus.fit_transform(yplus)[:,0]
# define output, Y (f is known)
Y = f
# Training loop
for epoch in range(max_no_epoch):
# Compute prediction and loss, L
    o = model(X) #prediction
    L = loss_fn(o, Y)  # L=|o-Y|_2
    L.backward()
```

- `loss.backward()` computes $dL/dw_1, dL/db_1, dL/ds_1, \ldots$
- They are used to get new improved $w_1, b_1, \ldots$

$$\left(\nu + \nu_{t,NN}\right) \frac{d^2 k_{DNS}}{dy^2} + \frac{dk_{DNS}}{dy}\frac{d\nu_{t,NN}}{dy} + P^k_{DNS} - \varepsilon_{DNS} = Q \qquad (3)$$

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def ODE(y, nut):
   nut_y = torch.autograd(nut, y, torch.ones(x.size()[0], 1,),create_graph=True)[0]
# Differential equation loss
   ODE_loss = (nu+nut)*k_yy + k_y*nut_y + Pk - eps
   ODE_loss = torch.sum(ODE_loss ** 2)
# b.c. loss
   BC_loss = (nut[0] - nut_0) ** 2
   return ODE_loss, BC_loss
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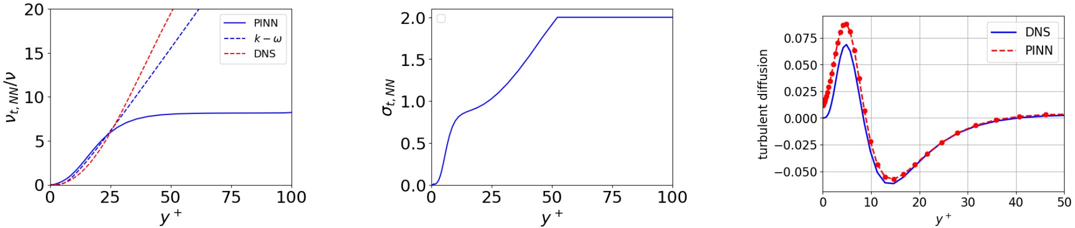
SOLVING EQ. 2 WITH PINN.



(A) Turbulent viscosity. (B) Prandtl number. (C) Turbulent diffusion.

FIGURE: k equation.

- Fully-developed flow in half a channel at $Re_\tau = 5\,200$.
- $\sigma_{t,NN} = \nu_t/\nu_{t,NN}$ (ν_t is the turbulent viscosity predicted by the Wilcox $k-\omega$ model)
- $\sigma_{t,NN}$ is limited to 2 (same as σ_k in the Wilcox $k-\omega$ model)

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CFD SOLVER

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- The discretized equations are solved with Python sparse matrix solvers.

The equation below is solved is using **pyCALC-RANS**

$$\frac{d}{dy}\left(\nu + \nu_{t,NN}\frac{dk}{dy}\right) + P^k_{DNS} - \varepsilon_{DNS} = 0$$

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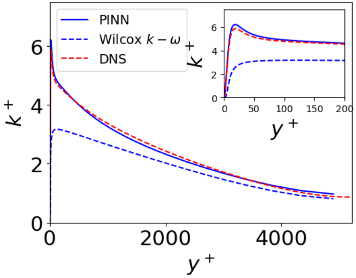

FIGURE: Turbulent kinetic energy.

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- Finally, the ω equation in the new $k-\omega$ model must predict $\omega = \omega_{DNS}$

$$\frac{d}{dy}\left(\frac{\nu_t}{\sigma_\omega}\frac{d\omega_{DNS}}{dy}\right) + C_{\omega 1}\frac{\omega_{DNS}}{k_{DNS}}P^k_{DNS} - C_{\omega 2}\omega^2_{DNS} = 0$$

PLOT C_k AND $C_{\omega 2}$

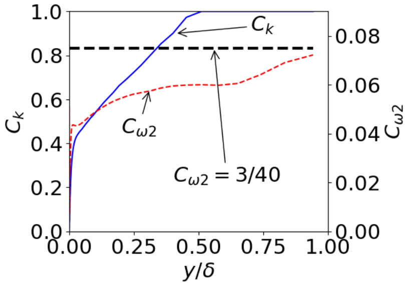
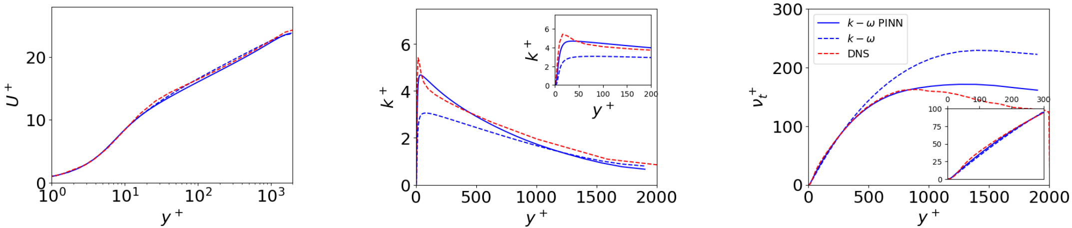


FIGURE: C_k and $C_{\omega 2}$ vs. y/δ.

RESULTS. CHANNEL FLOW. $Re_\tau = 2\,000$



(A) Velocity. (B) Turb. kinetic energy. (C) Turbulent viscosity.

FIGURE: Fully-developed channel flow. $Re_\tau = 2\,000$.

RESULTS. CHANNEL FLOW. $Re_\tau = 5\,200$

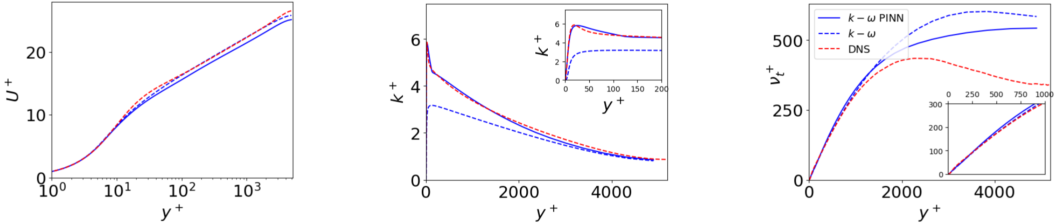


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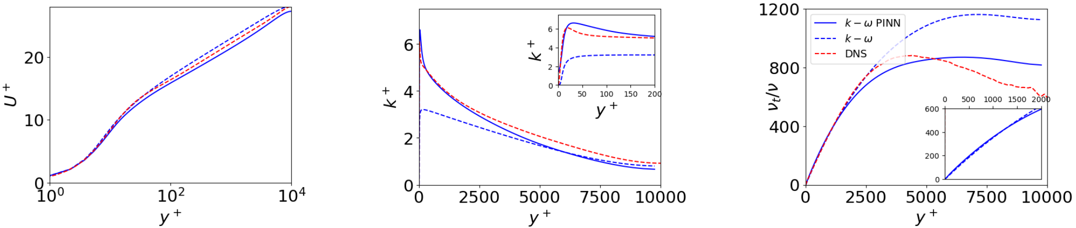
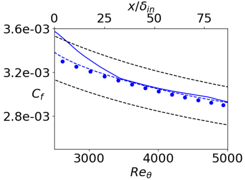
FIGURE: Fully-developed channel flow. $Re_\tau = 5\,200$.

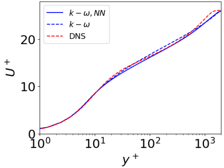


(A) Velocity. (B) Turb. kinetic energy. (C) Turbulent viscosity.

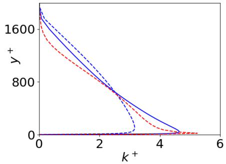
FIGURE: Fully-developed channel flow. $Re_\tau = 10\,000$.



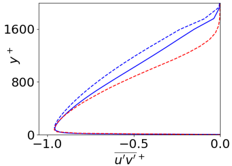
(A) Skin friction.



(B) Velocity.



(C) Turb. kinetic energy.



(D) Turbulent shear stress.

FIGURE: Flat-plate boundary layer. Profiles at $Re_\theta = 4\,500$.

- Inlet profiles from a pre-cursor RANS at $Re_\theta = 2\,500$

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- Hence, the current formulation of the model is not applicable to re-circulating flow
- Using Neural Network (NN), I've tried to make them functions of different input parameters such a P_k/ε, P_k^+, $\nu_t/(yu_\tau)$, ...

- I have made σ_k, C_K and $C_{\omega 2}$ functions of y/δ
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- Finally, I found a good combination input parameters: $\overline{u'v'}/u_\tau^2$ and $\nu_t/(yu_\tau)$ (not shown)

- The $k-\omega$ model has been modified using PINN so that it accurately predicts the turbulent kinetic energy

CONCLUDING REMARKS

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- You can download the ETMM15 paper, pyCALC-RANS and PINN scripts here or Google pyCALC-RANS PINN

- Neural Network and PINN in Python.
  - Good YouTube lectures: ”3Blue1Brown: But what is a neural network”; ”3Blue1Brown: gradient descent, how neural networks learn”; ”3Blue1Brown: backpropagation, intuitively”; ”3Blue1Brown: backpropagation, calculus”; ”Sebastian Lague: how to create a neural network”.

Download

- Download paper, Python scripts and CFD codes

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